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Quantum turbulence—from superfluid helium to atomic Bose–Einstein condensates

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Abstract
This paper reviews recent developments in the physics of quantum turbulence (QT). QT was discovered in superfluid ^{4}He in the 1950s, while the research has taken a new direction since the middle of the 1990s. QT is comprised of quantized vortices that are definite topological defects and expected to give a prototype of turbulence much simpler than usual classical turbulence. We give a general introduction and brief review of classical turbulence followed by a description of the dynamics of quantized vortices. After mentioning the modern research trends in QT, we discuss the energy spectra, the energy cascade and the possible dissipation mechanism of QT at very low temperatures. The last part is devoted to QT in atomic Bose–Einstein condensates.

1. Introduction

Turbulence has long been a great mystery in nature and has been studied in many fields over the centuries [1], but it is still not yet well understood. This is chiefly because turbulence is a complicated dynamical phenomenon with strong nonlinearity and is far from equilibrium. Leonardo da Vinci observed the turbulent flow of water and drew many sketches showing that turbulence had a structure comprised of eddies. However, these eddies are not well defined in a classical fluid and the relation between turbulence and vortices is not clear.

In the field of low temperature physics, turbulence in superfluid helium has been studied. Superfluid turbulence is often called quantum turbulence, because it is strongly influenced by quantum effects and comprised with quantized vortices. A quantized vortex is a definite topological defect appearing in a Bose–Einstein condensate, very different from an eddy in classical fluid. Quantum turbulence is now studied in superfluid ^{4}He, ^{3}He and even in atomic Bose–Einstein condensates (BECs). This paper reviews the recent developments and the understanding of this fascinating topic. This article cannot cover all modern topics of QT. For example, recent excellent developments in superfluid ^{3}He are not described here. For more comprehensive reviews, readers should refer to [2] and [3].

2. Superfluid helium and its turbulence

This section describes briefly superfluid helium, quantized vortices and the previous studies on quantum turbulence.

2.1. Superfluid helium and quantized vortices

Liquid ^{4}He enters a superfluid state below the λ point (2.17 K) with Bose–Einstein condensation of the ^{4}He atoms [4]. The hydrodynamics of superfluid helium is well described by the two-fluid model, for which the system consists of an inviscid superfluid (density ρ_s) and a viscous normal fluid (density ρ_n) with two independent velocity fields $\mathbf{v}_s$ and $\mathbf{v}_n$. The mixing ratio of the two fluids depends on temperature. As the temperature is reduced below the λ point, the ratio of the superfluid component increases, and the whole fluid becomes a superfluid below about 1 K. The Bose-condensed system exhibits the macroscopic wavefunction $\Psi(\mathbf{x}, t) = |\Psi(\mathbf{x}, t)|e^{i\theta(\mathbf{x},t)}$ as an order parameter. The superfluid velocity field is given by $\mathbf{v}_s = (\hbar/m)\nabla\theta$, with boson mass m representing the potential flow. Since the macroscopic wavefunction should be single-valued for the space coordinate $\mathbf{x}$, the circulation $\Gamma = \oint \mathbf{v} \cdot d\ell$ for an arbitrary closed loop in the fluid is quantized by the quantum $\kappa = h/m$. A vortex with such quantized circulation is called a quantized vortex. Any rotational motion of a superfluid is sustained only by quantized vortices.

A quantized vortex is a topological defect characteristic of a Bose–Einstein condensate and is different from a vortex in a classical viscous fluid. First, the circulation is quantized. Second, a quantized vortex is a vortex of inviscid superflow, being free from the decay mechanism of the viscous diffusion of vorticity that occurs in a classical fluid. Third, the core of a quantized vortex is very thin, of the order of the coherence length, which is only a few angstroms in superfluid ^{4}He and sub-μm in atomic BECs. These properties make a quantized vortex more stable and definite than a classical vortex.

2.2. Previous studies on quantum turbulence

Early experimental studies on superfluid turbulence chiefly focused on thermal counterflow, in which the normal fluid and superfluid flow in opposite directions. The flow is driven by injected heat current, and it was found that the superflow becomes dissipative when the relative velocity between the two fluids exceeds a critical value [5]. Feynman proposed that this is a superfluid turbulent state consisting of a tangle of quantized vortices [6]. Vinen later confirmed Feynman's proposition experimentally by showing that the dissipation comes from mutual friction between vortices and the normal flow [7–9]. Subsequently, many experimental studies have been made on superfluid turbulence (ST) in thermal counterflow systems, and have revealed much physics [10]. Since the dynamics of quantized vortices is nonlinear and nonlocal, it has not been easy to understand quantitatively the observations from vortex dynamics. Schwarz clarified the picture of ST consisting of tangled vortices by a numerical simulation of the quantized vortex filament model in thermal counterflow [11, 12]. ST is also often called quantum turbulence (QT), which emphasizes the fact that it is comprised of quantized vortices.

2.3. Dynamics of quantized vortices

Understanding the dynamics of quantized vortices is indispensable for revealing QT. Two formulations are generally available. One is the vortex filament model and the other is the Gross–Pitaevskii (GP) model. We will briefly describe these two formulations.

The vortex filament model represents a quantized vortex as a filament passing through the fluid, having a definite direction corresponding to its vorticity. Although this model has been used in classical fluid dynamics [13], it is only a convenient idealization. However, the model is accurate and realistic for a quantized vortex in superfluid helium. Except for the thin core region, the superflow velocity field has a classically well-defined meaning and can be described by ideal fluid dynamics [11, 14]. The superfluid velocity at a point on a filament is given by the Biot–Savart expression. The vortex moves with the superfluid velocity at its point. At finite temperatures mutual friction operates between the vortex core and the normal flow. Starting with several remnant vortices, Schwarz studied numerically how they developed to a statistical steady vortex tangle under thermal counterflow [12]. The tangle was self-sustained by the competition between the excitation due to the applied flow and the dissipation through mutual friction. The numerical results were quantitatively consistent with some typical experimental results. Here we shall introduce some quantities characteristic of a vortex tangle. The line length density L is defined as the total length of vortex cores in a unit volume. The mean spacing ℓ between vortices is given by $\ell = L^{-1/2}$.

The other is the Gross–Pitaevskii (GP) model. In a weakly interacting Bose system, the macroscopic wave-function $\Psi(\mathbf{x}, t)$ appears as the order parameter of Bose–Einstein condensation, obeying the Gross–Pitaevskii (GP) equation [15]:

$$\mathrm{i}\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + g|\Psi(\mathbf{x},t)|^2 - \mu\right)\Psi(\mathbf{x},t). \quad (1)$$

Here $g = 4\pi\hbar^2 m/a$ represents the strength of the interaction characterized by the s-wave scattering length a, m is the mass of each particle and μ is the chemical potential. The only characteristic scale of the GP model is the coherence length defined by $\xi = \hbar/(\sqrt{2mg}|\Psi|)$, which gives the vortex core size. The GP model can explain not only the vortex dynamics but also phenomena concerned with vortex cores such as reconnection and nucleation. However, the GP equation is not applicable quantitatively to superfluid ^{4}He, which is not a weakly interacting Bose system. It is applicable rather to Bose–Einstein condensation of a dilute atomic Bose gas [15].

3. Classical turbulence

Before going to the modern research on QT, we shall remember classical turbulence (CT) [1]. Classical viscous fluid dynamics is described by the Navier–Stokes equation. For high flow velocity or the Reynolds number, flow is generally turbulent, in which the flow is highly complicated with many eddies.

It is almost impossible and meaningless to follow the change of the fluid velocity in a turbulent flow and predict something; it becomes important to consider statistical laws instead of the dynamics of each variable. Turbulent flow is known to show some characteristic statistical behaviour [16, 17]. We suppose a steady state of fully developed turbulence of an incompressible classical fluid. The energy is injected into the fluid at a rate ε, which is at a scale comparable to the system size in the energy-containing range. In the inertial range, this energy is transferred to smaller scales without being dissipated. In this range, the system is locally homogeneous and isotropic, which leads to the statistics of the energy spectrum known as the Kolmogorov law:

$$E(k) = C\varepsilon^{2/3}k^{-5/3}. \quad (2)$$

Here, the energy spectrum $E(k)$ is defined as $E = \int \mathrm{d}\mathbf{k}\, E(k)$, where E is the kinetic energy per unit mass and k is the wavenumber from the Fourier transformation of the velocity field. The energy transferred to smaller scales in the energy-dissipative range is dissipated through the viscosity of the fluid with dissipation rate ε in equation (2), which is equal to the energy flux Π in the inertial range. The Kolmogorov constant C is a dimensionless parameter of order unity. The Kolmogorov spectrum is confirmed experimentally and numerically in turbulence with high Reynolds numbers. The inertial range is thought to be sustained by the self-similar Richardson cascade in which large eddies are broken up into smaller ones through many vortex reconnections. In CT, however, the Richardson cascade is not well fixed because it is impossible to definitely identify each eddy.

4. Towards modern research on QT

Most older experimental studies on QT were devoted to thermal counterflow. Since this flow has no classical analogue, these studies did not greatly contribute to the understanding of the relation between CT and QT. From the middle of 1990s, important experimental studies were published on QT not featuring thermal counterflow, differing significantly from previous studies [3, 18, 19].

The first important contribution was made by Maurer and Tabeling [20], who confirmed experimentally the Kolmogorov spectrum in superfluid ^{4}He for the first time. A turbulent flow was produced in a cylinder by driving two counter-rotating disks. The authors observed the local pressure fluctuations to obtain the energy spectrum. The experiments were made at three different temperatures 2.3, 2.08 and 1.4 K. Both above and below the λ point, the Kolmogorov spectrum was confirmed.

The next significant piece of work was a series of experiments of grid turbulence performed for superfluid ^{4}He above 1 K by the Oregon group [21–24]. Flow through a grid is usually used for generating turbulence in classical fluid dynamics [1]. At a sufficient distance behind the grid the flow displays a form of homogeneous isotropic turbulence. This method has also been applied to superfluid helium. In the Oregon experiments the helium was contained in a channel, along which a grid was pulled at a constant velocity. A pair of second-sound transducers was set into the walls of the channel to observe a vortex tangle. In combining the observations with the decay of the turbulence, the authors made some assumptions. The analysis is too complicated to be described in detail here. The thing is that the coupling between the superfluid and the normal fluid by mutual friction makes a quasi-classical flow appear at length scales much larger than ℓ and causes the fluid to behave like a one-component fluid [25]. Then the line length density is found to decay as $t^{-3/2}$. A simple analysis shows that the $t^{-3/2}$ decay comes from a quasi-classical model with the Kolmogorov spectrum [18].

After these developments, lots of experimental, theoretical and numerical works appeared in this field. The present main research areas into QT are as follows. The first is the energy spectra and the dissipation mechanism at zero temperature [26]. The second is QT created by vibrating structures [27]. The third is visualization of QT [28]. Since the main theme here is to develop the research on QT from superfluid helium to atomic BECs, we will focus on the first issue in the rest of the article.

5. Energy spectra of QT at zero temperature

What happens to QT at zero temperature is not so trivial [26]. The first problem is the nature of the energy spectrum of turbulence for the pure superfluid component. The second problem is the dissipation in this system. Since there is no normal fluid component, any dissipative mechanism does not work at large scales. However, some dissipative mechanism should operate at small scales. The first possibility is acoustic emission at vortex reconnections. In classical fluid dynamics it is known that vortex reconnections cause acoustic emission. In quantum fluids, numerical simulations of the GP model show acoustic emission at every reconnection event [29]. However, this mechanism is thought not to be important in superfluid helium because of the very short coherence length; it may be more efficient in atomic BECs in which the coherence length is generally not much shorter than the system size. The second possible mechanism is the radiation of sound (phonons) by the oscillatory motion of vortex cores. The third problem is how the energy is transferred from large to small scales where some dissipative mechanism operates. If the energy spectrum obeys the Kolmogorov law, the energy should be transferred through the *genuine* Richardson cascade of quantized vortices; this is very different from the classical case, because the Richardson cascade is only schematic in CT. Vortex reconnections sustaining the Richardson cascade should be less effective at scales shorter than the mean vortex spacing ℓ. What transfers the energy there? These problems are closely related and are described below.

5.1. Kolmogorov law in the inertial range

No experimental studies have yet addressed this issue directly, though three numerical studies have been made to date. The first study was done by Nore *et al* using the GP model [30, 31]. They solved numerically the GP equation starting from Taylor–Green vortices and followed their time development. The quantized vortices become tangled and the energy spectra of the incompressible kinetic energy seemed to obey the Kolmogorov law for a short period, though eventually deviating from it. The second study was done by the vortex filament model [32]. Araki *et al* made a vortex tangle arising from Taylor–Green vortices and obtained an energy spectrum consistent with the Kolmogorov law. The third was by the modified GP model [33, 34]. This should be most relevant to atomic BECs, described here.

The Kolmogorov spectra were confirmed for both decaying [33] and steady [34] QT by the modified GP model. The normalized GP equation is

$$\mathrm{i}\frac{\partial}{\partial t}\Phi(\mathbf{x},t) = [-\nabla^2 - \mu + g|\Phi(\mathbf{x},t)|^2]\Phi(\mathbf{x},t), \qquad (3)$$

which determines the dynamics of the macroscopic wavefunction $\Phi(\mathbf{x},t) = f(\mathbf{x},t)\exp[\mathrm{i}\phi(\mathbf{x},t)]$. The condensate density is $|\Phi(\mathbf{x},t)|^2 = f(\mathbf{x},t)^2$ and the superfluid velocity $\mathbf{v}(\mathbf{x},t)$ is given by $\mathbf{v}(\mathbf{x},t) = 2\nabla\phi(\mathbf{x},t)$. The vorticity $\omega(\mathbf{x},t) = \mathrm{rot}\mathbf{v}(\mathbf{x},t)$ vanishes everywhere in a single-connected region of the fluid and thus every rotational flow is carried by quantized vortices. In the core of each vortex, $\Phi(\mathbf{x},t)$ vanishes so that the circulation around the core is quantized by 4π. The vortex core size is given by the healing length $\xi = 1/f\sqrt{g}$.

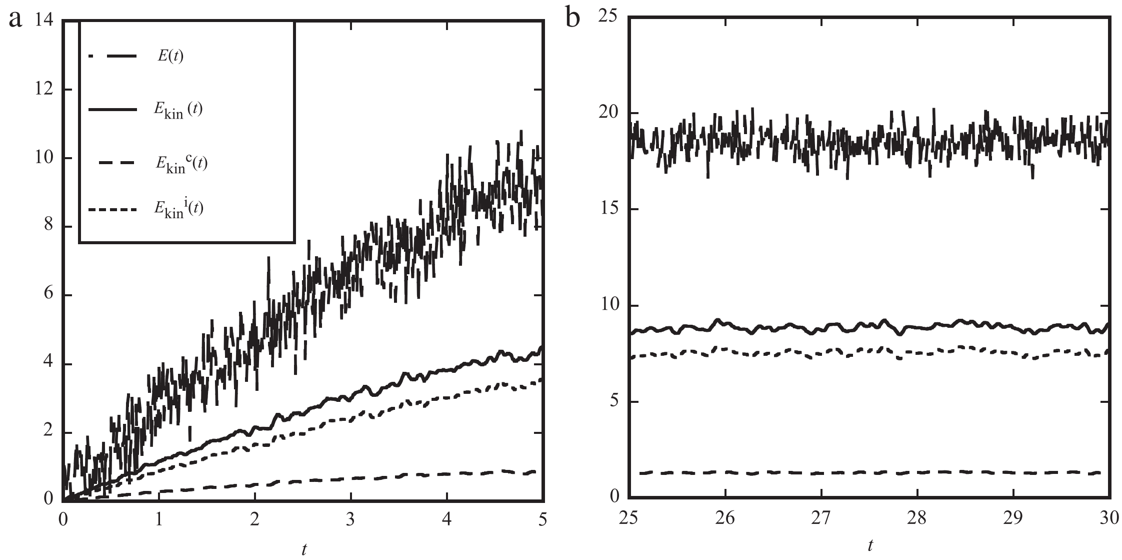

Figure 1. Time development of $E(t)$, $E_{\rm kin}(t)$, $E^{\rm c}_{\rm kin}(t)$ and $E^{\rm i}_{\rm kin}(t)$ at (a) the initial stage $0 \leqslant t \leqslant 5$ and (b) a later stage $25 \leqslant t \leqslant 30$ [34].

It should be noted that the hydrodynamics by the GP model is compressible. The total energy is

$$E(t) = \frac{1}{N}\int \mathrm{d}\mathbf{x}\,\Phi^*(\mathbf{x},t)\Big[-\nabla^2 + \frac{g}{2}f(\mathbf{x},t)^2\Big]\Phi(\mathbf{x},t), \quad (4)$$

which is represented by the sum of the interaction energy $E_{\rm int}(t)$, the quantum energy $E_{\rm q}(t)$ and the kinetic energy $E_{\rm kin}(t)$ [30, 31]:

$$E_{\rm int}(t) = \frac{g}{2N}\int \mathrm{d}\mathbf{x}\, f(\mathbf{x},t)^4,$$
$$E_{\rm q}(t) = \frac{1}{N}\int \mathrm{d}\mathbf{x}\,[\nabla f(\mathbf{x},t)]^2, \quad (5)$$
$$E_{\rm kin}(t) = \frac{1}{N}\int \mathrm{d}\mathbf{x}\,[f(\mathbf{x},t)\nabla\phi(\mathbf{x},t)]^2.$$

The kinetic energy is furthermore divided into a compressible part $E^{\rm c}_{\rm kin}(t)$ due to compressible excitations and an incompressible part $E^{\rm i}_{\rm kin}(t)$ due to vortices. The Kolmogorov spectrum could be expected for $E^{\rm i}_{\rm kin}(t)$.

The failure to obtain the Kolmogorov law for the pure GP model [30, 31] could be attributable to the following reasons. Their simulation showed that $E^{\rm i}_{\rm kin}(t)$ decreased and $E^{\rm c}_{\rm kin}(t)$ increased with conserving the total energy $E(t)$. This was because many compressible excitations were created through vortex reconnections [29, 35] and disturbed the Richardson cascade of quantized vortices. Kobayashi and Tsubota overcame the difficulties and obtained the Kolmogorov spectra in QT to clearly reveal the energy cascade [33, 34]. They made a numerical calculation for the Fourier transformed GP equation with dissipation:

$$(\mathrm{i} - \tilde{\gamma}(\mathbf{k}))\frac{\partial}{\partial t}\tilde{\Phi}(\mathbf{k},t) = [k^2 - \mu(t)]\tilde{\Phi}(\mathbf{k},t)$$
$$+ \frac{g}{V^2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\tilde{\Phi}(\mathbf{k}_1,t)\tilde{\Phi}^*(\mathbf{k}_2,t)\times\tilde{\Phi}(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2,t). \quad (6)$$

Here $\tilde{\Phi}(\mathbf{k},t)$ is the spatial Fourier component of $\Phi(\mathbf{x},t)$ and V is the system volume. The healing length is given by $\xi = 1/|\Phi|\sqrt{g}$. The dissipation should have the form $\tilde{\gamma}(\mathbf{k}) = \gamma_0\theta(k - 2\pi/\xi)$ with the step function θ, which dissipates only the excitations smaller than ξ. This form of dissipation can be justified by the coupled analysis of the GP equation and the Bogoliubov–de Gennes equations for thermal excitations [36].

First Kobayashi *et al* confirmed the Kolmogorov spectra for decaying turbulence [33]. To obtain a turbulent state, they started the calculation from an initial configuration in which the density was uniform and the phase of the wavefunction had a random spatial distribution. The initial state was dynamically unstable and soon developed into turbulence with many vortex loops. Then the spectrum $E^{\rm i}_{\rm kin}(k,t)$ was found to obey the Kolmogorov law.

A more elaborate analysis was made for steady QT by introducing energy injection at large scales as well as energy dissipation at small scales [34]. Energy injection at large scales was effected by moving a random potential $V(\mathbf{x},t)$ that had a characteristic spatial scale X_0. Once a steady QT is made by the balance between the energy injection and the dissipation, it should have an energy-containing range $k < 2\pi/X_0$, inertial range $2\pi/X_0 < k < 2\pi/\xi$ and energy-dissipative range $2\pi/\xi < k$. A typical simulation of steady turbulence was performed for $X_0 = 4$. The dynamics started from the uniform wavefunction. Figure 1 shows the time development of each energy component. The moving random potential nucleates sound waves as well as vortices, but both figures show that the incompressible kinetic energy $E^{\rm i}_{\rm kin}(t)$ due to vortices is dominant in the total kinetic energy $E_{\rm kin}(t)$. The four energies are almost constant for $t \geqslant 25$ and steady QT was obtained.

The cascade can be confirmed quantitatively by checking that the energy dissipation rate ε of $E^{\rm i}_{\rm kin}(t)$ is comparable to the flux of energy $\Pi(k,t)$ through the Richardson cascade in the inertial range. The energy flux $\Pi(k,t)$ is found to be approximately independent of k and comparable to ε. As shown in figure 2(b), the energy spectrum is quantitatively

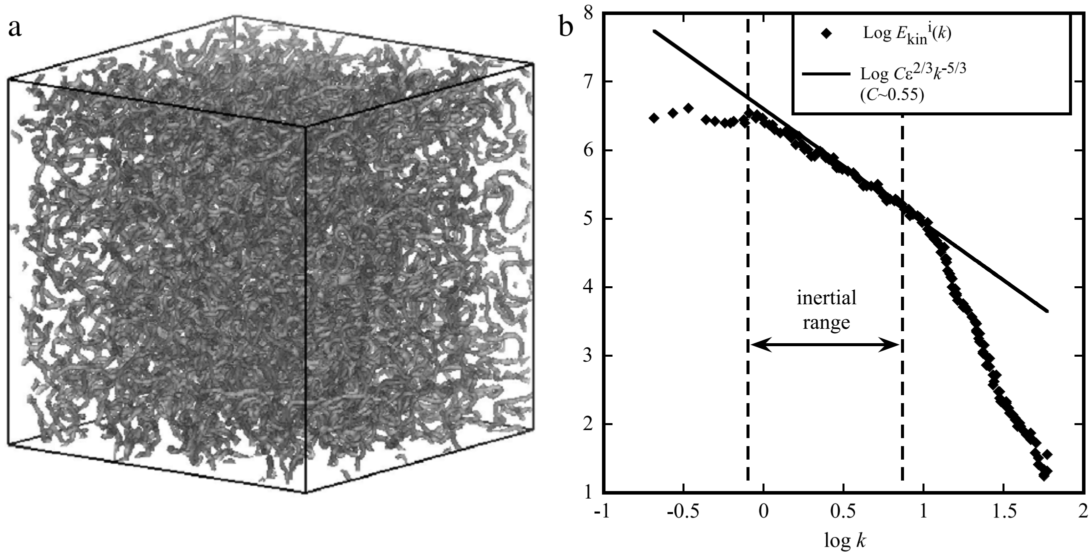

Figure 2. (a) A typical vortex tangle. (b) Energy spectrum $E^{\mathrm{i}}_{\mathrm{kin}}(k,t)$ for QT. The plotted points are from an ensemble average of 50 randomly selected states at $t > 25$. The solid line is the Kolmogorov law [34].

consistent with the Kolmogorov law in the inertial range $2\pi/X_0 < k < 2\pi/\xi$, which is equivalent to $0.79 < k < 6.3$.

5.2. The Kelvin-wave cascade

The arguments in the last subsection were chiefly limited to the large scale, usually larger than the mean spacing ℓ of a vortex tangle, in which the Richardson cascade is effective. Here we should ask what happens at smaller scales for which the Richardson cascade should be less effective. The most probable scenario is the Kelvin-wave cascade. A Kelvin-wave is a deformation of a vortex line into a helix with the deformation propagating as a wave along the vortex line [37]. Kelvin-waves were first observed by making torsional oscillations in uniformly rotating superfluid ^{4}He [38, 39]. The approximate dispersion relation for a rectilinear vortex is $\omega_k = (\kappa k^2)/(4\pi)(\ln(1/ka_0) + c)$ with a constant $c \sim 1$. This k is a wavenumber of an excited Kelvin-wave, being different from the wavenumber used for the energy spectrum in the last subsection. At a finite temperature a significant fraction of normal fluid damps Kelvin-waves through mutual friction. At very low temperatures, however, mutual friction does not occur and the only possible mechanism of dissipation is radiation of phonons [40]. Phonon radiation becomes effective only when the frequency becomes very high, typically of the order of GHz ($k \sim 10^{-1}$ nm^{-1}), so a mechanism is required to transfer the energy to such high wavenumbers for Kelvin-waves to damp. An early numerical simulation based on the vortex filament model showed that Kelvin-waves are unstable to the buildup of side bands [41]. This indicates the possibility that nonlinear interactions between different Kelvin-wavenumbers can transfer energy from small to large wavenumbers, namely the Kelvin-wave cascade. This idea was first suggested by Svistunov [42] and later developed and confirmed through theoretical and numerical works by Vinen, Tsubota and Mitani [43] and Kozik and Svistunov [44–46].

It is difficult to observe the Kelvin-wave cascade. Such studies are not easy for a vortex tangle. The easiest approach would be to consider rotation. In a rotating vessel, quantized vortices form a vortex lattice parallel to the rotational axis. By oscillating the vortices, Kelvin-waves can be excited. The challenge is detecting the Kelvin-wave cascade. There are two possible methods. The first is the direct visualization of vortices. Recently Bewley *et al* visualized quantized vortices by trapping micron-sized solid hydrogen particles [47]. They also observed a vortex array under rotation. The direct observation of vortex dynamics could reveal the Kelvin-wave cascade. The second method is to observe acoustic emission resulting from a Kelvin-wave cascade. Since the frequency of emitted phonons is estimated to be of GHz order, this observation may not be easy and it presents a challenging experimental problem. Observing the Kelvin-wave cascade may be more accessible in atomic BECs.

It is important to ask the nature of the transition between the Richardson (classical) and the Kelvin-wave (quantum) cascades. Several theoretical considerations on the classical-quantum crossover have been made [48, 49], yet there are few numerical or experimental works. This topic is not yet fixed, and is still controversial, so not discussed here.

6. QT in atomic BECs

The achievement of Bose–Einstein condensation in trapped atomic gases in 1995 stimulated intense experimental and theoretical activity in modern physics [15]. As a proof of superfluidity, quantized vortices were created and observed in atomic BECs, and lots of effort has been devoted to this fascinating problem [50]. Atomic BECs have several advantages over superfluid helium. First, the modern optical technique enables us to visualize directly quantized vortices. Secondly, the system is a weakly interacting Bose gas, which makes the GP model quantitatively correct. For example, the

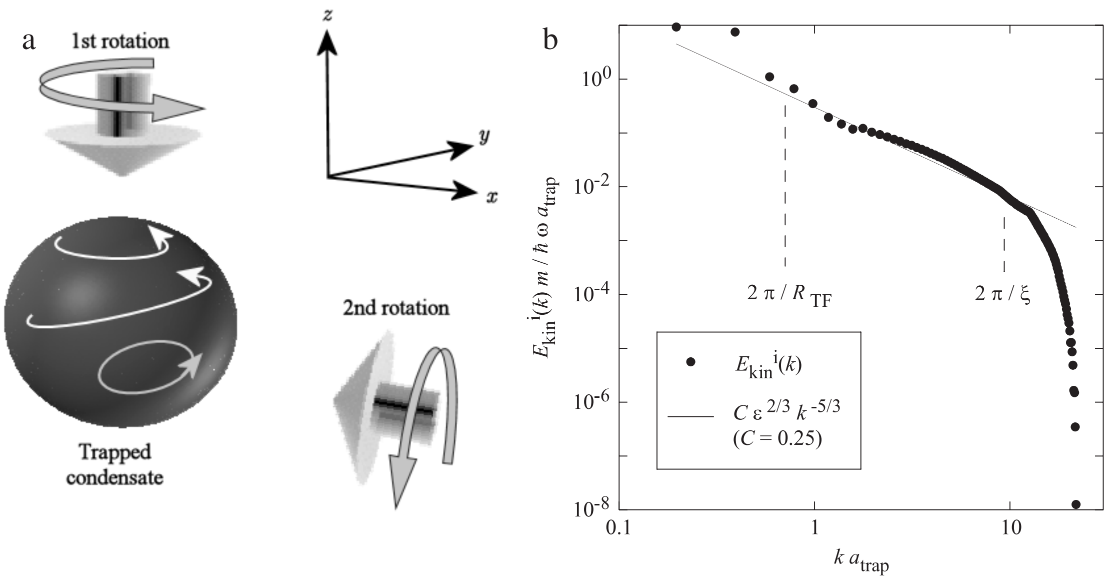

Figure 3. QT in atomic BECs. (a) Rotating the condensate around two axes. (b) Energy spectrum of a steady QT. The dots refer to the numerically obtained spectrum, while the solid line is the Kolmogorov spectrum [54].

observation of the dynamics of a vortex lattice formation in a rotating BEC [51] is well described by the GP model [52]. However, the studies of quantized vortices in cold atoms have not yet been so exhaustive, which can be seen easily by looking back over what has been studied in superfluid helium. The long research history of superfluid helium tells us two main cooperative phenomena of quantized vortices; the first is a vortex lattice in rotating superfluid and the other is a vortex tangle in QT. To date most studies of quantized vortices in atomic BECs have been limited to the former, and only a few for the latter. Recently it has been shown theoretically that QT can be created also in trapped BECs and the energy spectrum obeys the Kolmogorov law [53, 55]. This section summarizes the topic.

A problem is how to make QT in trapped BECs; the method should be experimentally accessible. Although a few methods were proposed initially [56, 57], Kobayashi and Tsubota suggested an easier and more powerful method in order to make a steady QT in trapped BECs, namely combined rotations around two axes. The dynamics of the wavefunction is described by the GP equation

$$[\mathrm{i} - \gamma(\mathbf{x})]\hbar \frac{\partial}{\partial t}\Phi(\mathbf{x}, t) = \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + g|\Phi(\mathbf{x}, t)|^2 + U(\mathbf{x}) - \Omega(\mathbf{x}) \cdot \mathbf{L}(\mathbf{x}, t) \right]\Phi(\mathbf{x}, t). \quad (7)$$

Here $\mathbf{L}(\mathbf{x}) = \mathrm{i}\hbar\mathbf{x} \times \nabla$ is the angular momentum and the trapping potential $U(\mathbf{x})$ is given by a weakly elliptical harmonic potential:

$$U(\mathbf{x}) = \frac{m\omega^2}{2}[(1 - \delta_1)(1 - \delta_2)x^2 + (1 + \delta_1)(1 - \delta_2)y^2 + (1 + \delta_2)z^2], \quad (8)$$

where ω is the frequency of the harmonic trap and the parameters δ_1 and δ_2 exhibit elliptical deformation in the xy-and zx-planes. To develop the BEC into a turbulent state, two rotations along the z- and x-axes are applied, as shown in figure 3(a). The rotation vector $\Omega(t)$ is given by $\Omega(t) = (\Omega_x, \Omega_z \sin \Omega_x t, \Omega_z \cos \Omega_x t)$, where Ω_z and Ω_x are the frequencies of the first and second rotation, respectively. Note that this is not a simple sum of two rotations. Actually this sort of rotation is used for turbulence in water [58].

Starting from a stationary solution without rotation and elliptical deformation, Kobayashi *et al* turned on the rotation $\Omega_x = \Omega_z = 0.6$ and elliptical deformation $\delta_1 = \delta_2 = 0.025$, and numerically calculated the time development of the GP equation (7). By monitoring the total compressible and incompressible kinetic energy per unit mass and some anisotropic parameters, the system is found to become almost steady and isotropic after $\omega t \simeq 150$. For the steady state, the spectrum $E^{\mathrm{i}}_{\mathrm{kin}}(k, t)$ was calculated to be consistent with the Kolmogorov law (figure 3(b)). The inertial range is determined by the Thomas–Fermi radius R_{TF} and the coherence length ξ. The application of combined rotation around three axes enables us to obtain more isotropic QT [55]. Here we should worry about how to observe the energy spectrum. The technique of Bragg spectroscopy [59] would enable us to observe the local superfluid velocity in the condensate and thus the energy spectrum.

There are several advantages of studying QT in atomic BECs instead of in superfluid helium. First, it is possible to observe the vortex configuration, probably even the Richardson cascade. There are a few theoretical or numerical works which show some power law of the vortex size distribution characteristics of the Richardson cascade [32, 60, 61], there is no experimental proof though. The research in this direction is quite important, because we can study directly the relation between the real-space Richardson cascade and the wavenumber-space cascade (the Kolmogorov spectrum). Secondly, we can control the transition to turbulence by changing the rotation frequencies or other parameters. For example, we know that rotation along one axis forms a vortex lattice. When we apply another rotation, it may just rotate the lattice if the frequency is low. If the frequency is high enough, however, the second rotation should destroy the lattice towards a vortex tangle. It would be possible to investigate in detail the transition to turbulence. Thirdly, by changing the shape of the trapping potential, we can study the effect of the anisotropy on turbulence; a typical question is how the Kolmogorov spectrum is changed when the BEC becomes anisotropic. This interest should lead to studies of the transition between 2D and 3D turbulence.

7. Conclusions

This article has reviewed recent developments in the research of quantum turbulence. The research of QT now ranges from traditional superfluid helium to atomic BECs. These interdisciplinary studies will undoubtedly make a breakthrough in solving the great mystery of *turbulence* in nature which researchers since Leonard Da Vinci have been investigating. Furthermore the study of QT could also contribute to cosmology, because the evolution of cosmic strings in the early universe may be related to QT [62].

References

[1] Frisch U 1995 *Turbulence* (Cambridge: Cambridge University Press)
[2] Halperin W P and Tsubota M (ed) 2008 *Progress in Low Temperature Physics* vol 16 (Amsterdam: Elsevier)
[3] Tsubota M 2008 *J. Phys. Soc. Japan* **77** 111006
[4] Tilley D R and Tilley J 1990 *Superfluidity and Superconductivity* 3rd edn (Bristol: Institute of Physics Publishing)
[5] Gorter C J and Mellink J H 1949 *Physica* **15** 285
[6] Feynman R P 1955 *Progress in Low Temperature Physics* vol 1, ed C J Gorter (Amsterdam: North-Holland) p 17
[7] Vinen W F 1957 *Proc. R. Soc.* A **240** 114
[8] Vinen W F 1957 *Proc. R. Soc.* A **240** 128
[9] Vinen W F 1957 *Proc. R. Soc.* A **242** 493
[10] Tough J T 1982 *Progress in Low Temperature Physics* vol 8, ed C J Gorter (Amsterdam: North-Holland) p 133
[11] Schwarz K W 1985 *Phys. Rev.* B **31** 5782
[12] Schwarz K W 1988 *Phys. Rev.* B **38** 2398
[13] Saffman P G 1992 *Vortex Dynamics* (Cambridge: Cambridge University Press)
[14] Tsubota M, Araki T and Nemirovskii S K 2000 *Phys. Rev.* B **62** 11751
[15] Pethick C J and Smith H 2002 *Bose–Einstein Condensation in Dilute Gases* (Cambridge: Cambridge University Press)
[16] Kolmogorov A N 1941 *Dokl. Akad. Nauk SSSR* **30** 301
Kolmogorov A N 1991 *Proc. R. Soc.* A **434** 9 (reprinted)
[17] Kolmogorov A N 1941 *Dokl. Akad. Nauk SSSR* **31** 538
Kolmogorov A N 1991 *Proc. R. Soc.* A **434** 15 (reprinted)
[18] Vinen W F and Niemela J J 2002 *J. Low Temp. Phys.* **128** 167
[19] Vinen W F 2006 *J. Low Temp. Phys.* **145** 7
[20] Maurer J and Tabeling P 1998 *Europhys. Lett.* **43** 29
[21] Stalp S R, Skrbek L and Donnelly R J 1999 *Phys. Rev. Lett.* **82** 4831
[22] Skrbek L, Niemela J J and Donnelly R J 2000 *Phys. Rev. Lett.* **85** 2973
[23] Skrbek L and Stalp S R 2000 *Phys. Fluids* **12** 1997
[24] Stalp S R, Niemela J J, Vinen W F and Donnelly R J 2002 *Phys. Fluids* **14** 1377
[25] Vinen W F 2000 *Phys. Rev.* B **61** 1410
[26] Tsubota M and Kobayashi M 2008 *Progress in Low Temperature Physics* vol 16, ed W P Halperin and M Tsubota (Amsterdam: Elsevier) p 1
[27] Skrbek L and Vinen W F 2008 *Progress in Low Temperature Physics* vol 16, ed W P Halperin and M Tsubota (Amsterdam: Elsevier) p 195
[28] Van Sciver S W and Barenghi C F 2008 *Progress in Low Temperature Physics* vol 16, ed W P Halperin and M Tsubota (Amsterdam: Elsevier) p 247
[29] Leadbeater M, Winiecki T, Samuels D C, Barenghi C F and Adams C S 2001 *Phys. Rev. Lett.* **86** 1410
[30] Nore C, Abid M and Brachet M E 1997 *Phys. Rev. Lett.* **78** 3296
[31] Nore C, Abid M and Brachet M E 1997 *Phys. Fluids* **9** 2644
[32] Araki T, Tsubota M and Nemirovskii S K 2002 *Phys. Rev. Lett.* **89** 145301
[33] Kobayashi M and Tsubota M 2005 *Phys. Rev. Lett.* **94** 065302
[34] Kobayashi M and Tsubota M 2005 *J. Phys. Soc. Japan* **74** 3248
[35] Ogawa S, Tsubota M and Hattori Y 2002 *J. Phys. Soc. Japan* **71** 813
[36] Kobayashi M and Tsubota M 2006 *Phys. Rev. Lett.* **97** 145301
[37] Thomson J 1880 *Phil Mag.* **10** 155
[38] Hall H E 1958 *Proc. R. Soc.* A **245** 546
[39] Hall H E 1960 *Phil. Mag. Suppl.* **9** 89
[40] Vinen W F 2001 *Phys. Rev.* B **64** 134520
[41] Samuels D C and Donnelly R J 1990 *Phys. Rev. Lett.* **64** 1385
[42] Svistunov B V 1995 *Phys. Rev.* B **52** 3647
[43] Vinen W F, Tsubota M and Mitani M 2003 *Phys. Rev. Lett.* **91** 135301
[44] Kozik E and Svistunov B V 2004 *Phys. Rev. Lett.* **92** 035301
[45] Kozik E and Svistunov B V 2005 *Phys. Rev. Lett.* **94** 025301
[46] Kozik E and Svistunov B V 2005 *Phys. Rev.* B **72** 172505
[47] Bewley G P, Lathrop D P and Sreenivasan K R 2006 *Nature* **441** 588
[48] L'vov V S, Nazarenko S V and Rudenko O 2007 *Phys. Rev.* B **76** 024520
[49] Kozik E and Svistunov B V 2008 *Phys. Rev.* B **77** 060502
[50] Kasamatsu K and Tsubota M 2008 *Progress in Low Temperature Physics* vol 16, ed W P Halperin and M Tsubota (Amsterdam: Elsevier) p 351
[51] Madison K W, Chevy F, Bretin V and Dalibard J 2001 *Phys. Rev. Lett.* **86** 4443
[52] Kasamatsu K, Tsubota M and Ueda M 2003 *Phys. Rev.* A **67** 033610
[53] Kobayashi M and Tsubota M 2007 *Phys. Rev.* A **76** 045603
[54] Tsubota M and Kobayashi M 2008 *J. Low Temp. Phys.* **150** 402
[55] Kobayashi M and Tsubota M 2008 *J. Low Temp. Phys.* **150** 587
[56] Svistunov B 2001 *Quantized Vortex Dynamics and Superfluid Turbulence* ed C F Barenghi, R J Donnelly and W F Vinen (Berlin: Springer) p 327
[57] Parker N G and Adams C S 2005 *Phys. Rev. Lett.* **95** 145301
[58] Goto S, Ishii N, Kida S and Nishioka M 2007 *Phys. Fluids* **19** 061705
[59] Stenger J, Inouye S, Chikkatur A P, Stamper-Kurn D M, Pritchard D E and Ketterle W 1999 *Phys. Rev. Lett.* **82** 4569
[60] Nemirovskii S K 2006 *Phys. Rev. Lett.* **96** 015301
[61] Mitani A and Tsubota M 2006 *Phys. Rev.* B **74** 024526
[62] Volovik G 2003 *The Universe in a Helium Droplet* (New York: Oxford University Press)